Lorentz Invariant -Et+px in Quantum Free Particle Wavefunction

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In previous notes (1)(2), we argued one may write two flow/flux equations for a free particle using an unknown function A(x,t) i.e. d/dx (partial) A = E(v) v ((1a)) and d/dt (partial) A = -E(v) ((1b)) where E(v) is energy which is a function of rest mass and velocity v=constant. A(x,t), but after taking derivatives, x/t is set to v. As shown in (2), one may write A=tL(v) and find E(v)=mo/sqrt(1-vv) (c=1) (and p=Ev) i.e. the result of special relativity. Thus special relativity seems to follow directly from a flow/flux equation where motion v increases energy. Alternatively, one may obtain both relativistic/nonrelativistic quantum mechanics for a free particle, by converting ((1a)) and ((1b)) into differential equations i.e. id/dt exp(iA) = Eexp(iA) and -id/dx exp(iA) = pexp(iA) with A=-Et+px yielding the same result. In both cases, physical motion v has physical implications. For special relativity, energy changes from mo to mo/sqrt(1-vv) and there are effects on clocks and lengths. For quantum mechanics there is time and space resolution linked to dx→1/p and dt→1/E.

In this note we try to investigate if there is a connection between the two. In particular we try to see why -Et+px within the free particle wavefunction exp(-iEt+px) is a Lorentz invariant.

Link Between Quantum Mechanics and Special Relativity

In previous notes (1), (2) we argued that one may obtain relativistic and nonrelativistic quantum mechanics in addition to the specific form of energy (and hence momentum too) for special relativity from two flow/flux equations for a free particle:

d/dx (partial) A(x,t) = p ((1a)) and d/dt (partial) A(x,t) = -E ((1b))

where A(x,t) is an unknown function. One may show in general (2) that A=t L(v) from which using p=E(v), ((1a)) and ((1b)) lead to: dL/dv = p and E= pv-L standard results from Lagrangian theory. Combining these with p=Ev and solving for E(v) yields mo/sqrt(1-vv) where c=1. Alternatively, ((1a)) and ((1b)) may be written as eigenfunction equations:

-i d/dx (partial) exp(i A) = p exp(i A) ((2a)) and i d/dt partial exp(iA) = E exp(iA) ((1b))

Thus exp(iA) is equivalent to exp(-iEt+ipx). ((3)). As a result, special relativity and quantum mechanics for a free particle follow from the same two flow equations ((1a)) and ((1b)). In both cases,motion leads to physical consequences. For special relativity, energy is mo/sqrt(1-vv) (c=1) i.e. motion is linked to extra energy which we argue is physical i.e. not just a math construct. For quantum mechanics, one has space and time resolution from exp(-iEt+ipx) i.e. dx→ 1/p and dt→1/E. Is there a connection between these two theories?

Quantum Mechanics and Special Relativity

We have argued that the quantum free particle solution exp(-iEt+px) follows from ((1a)) and ((1b)). These two equations also yield special relativity results E=mo/sqrt(1-vv) and p=Ev from which one may obtain the equation:

EE = pp + momo   ((3))

exp(-iEt+px) is a solution of ((3)) with E-> id/dt and p->-id/dx which is not a surprise because both follow from ((1a)) and ((1b)). ((3)), however, is of the form of a invariance equation i.e.

-EE + pp = -momo ((4))

Here mo is rest mass, but the LHS may contain any E(v) and p=Ev. Thus, there seems to be a dot product rule for the vector (p,E) namely ((4)). Given that ((3)) is an invariant equation (Lorentz invariant), one would expect its solution to also be invariant Thus, one might suspect that (x,t) is also a vector which follows the same dot product rule i.e.

-tt+xx = constant ((5))   -Et+px is the dot product of the two vectors (p,E) and (x,t)

((5)) implies, if one uses x=vt that: tt(1-vv) = constant or t→ to/sqrt(1-vv)
where to is the rest frame time. Thus a time interval measured in the rest frame is to/sqrt(1-vv) in the lab frame, i.e. it is a bigger value, the clock has slowed down in the moving frame.

Thus, we argue that it is no accident that -Et+px appears in the relativistic quantum free particle solution. It is not just a wave type solution.

Physical Interpretation of E(v) linked to Time in Quantum Mechanics

For a quantum bound system, the solution is exp(-iEnt)Wn(x) where Wn(x)=Sum over p a(p)exp(ipx). In such a case, En is the binding energy which together with (mo+M) (e.g electron and proton core) gives the rest mass of the system. This whole system may undergo motion v in which case, the special relativistic scenario changes the system rest mass (mo+M+En) by the factor 1/sqrt(1-vv). En, however, seems to have physical meaning because it is part of the bound system solution. In a previous note we argued that 1/En is proportional to cycling time within a quantum system. As an example, consider the time-independent Schrodinger equation with V(x) written as Sum over k Vk exp(ikx). Collecting the coefficients of exp(ipx) yields for any (p):

a(p)pp/2m + Sum over k Vk a(p-k) = En a(p) ((6))

Thus a cycling occurs in which an a(p-k) exp(i (p-k)x) combines with an exp(ikx) to form an exp(ipx). We argue that this full cycling takes a time proportional to 1/E for any p. Thus in a moving system En→ En/sqrt(1-vv) (c=1) and the cycling time (which is a physical process in the sense that a particle may be knocked out of the system with momentum p and have wavefunction exp(ipx)) changes according to special relativity.

One has to be careful, however, because this time resolution or cycling time changes in the opposite way to the Lorentz result: Tmoving = Tlab/sqrt(1-vv). In other words, 1/En original → sqrt(1-vv) / En original.

Conclusion

In conclusion, we argue that the Lorentz invariant quantum free particle solution exp(-iEt+ipx) follows from the two flow/flux equations: d/dx partial A(x,t) =p and d/dt (partial) A=-E written as eigenvalue equations. Furthermore these two flux equations yield special relativistic solutions E=mo/sqrt(1-vv) and p=Ev from which -EE+pp = momo follows. This has the form of an invariant or dot product equation and also has the solution exp(-iEt+ipx). Thus -Et+px should be an invariant i.e. a dot product. In such a case -tt+xx = constant and if x=vt then, t=to/sqrt(1-vv) which is the result from special relativity. There is a quantum time resolution value proportional to 1/E and this behaves in the opposite manner i.e 1/E → sqrt(1-vv) 1/E.

References

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2. Ruggeri, Francesco R. Ruggeri Special Relativity From Nonrelativistic Lagrangian Formalism? (preprint, zenodo, 2021)